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THREE-DIMENSIONAL ASYMPTOTIC FINITE ELEMENT METHOD FOR ANISOTROPIC INHOMOGENEOUS AND LAMINATED PLATES

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Abstract—An asymptotic finite element model for anisotropic inhomogeneous and laminated plates is developed within the framework of three-dimensional elasticity. The formulation begins with a Hellinger–Reissner functional in which the displacements and transverse stresses are taken to be the functions subject to variation. By means of asymptotic expansion the H–R functional for the problem is decomposed into functionals of various orders from which the asymptotic finite element equations are derived. In the multilevel computations the transverse stresses and displacements may be interpolated independently, and the midplane displacements are the only unknown nodal degree-of-freedoms (DOF) in the system equations, thus the total DOF at each level is less than that of a homogeneous Kirchhoff plate. The stiffness matrix remains unchanged; the one generated at the leading-order level is always used at subsequent levels. The formulation is three-dimensional yet requires only two-dimensional finite element discretization with no need of interpolation in the thickness direction. The through-thickness effect can be accounted for in a consistent and hierarchical manner. Numerical comparisons with the benchmark solutions show that the method is effective in modeling of multilayered composite plates.

1. INTRODUCTION

Numerical modeling of multilayered composite plates has been under continual study and there accumulates a vast amount of literature on the subject. A comprehensive review of different approaches to the problem can be found in a review paper by Noor and Burton (1989). Early development ranges from those based on two-dimensional classical laminated plate theory (CLT) (see, e.g., Whitney, 1987) to those based on higher-order theories (Lo *et al.*, 1977a; b; Reddy, 1984), in which variations of the displacements through the thickness are assumed *a priori* and the degree-of-freedom (DOF) at a node in the finite element model is assigned accordingly. The computational models based on CLT or its variants serve the purpose in providing reasonable predictions on structural responses of laminated plates with simple and straightforward computation. Yet it has been well recognized that the displacement models fail to yield reliable results on interlaminar stresses, especially when the plate is not thin. Many finite element models that take into account the through-thickness heterogeneities of multilayered plates have been proposed, including hybrid models (e.g., Mau *et al.*, 1972; Atluri *et al.*, 1983; Spilker, 1982). The hybrid models for the problem extend the displacement models by incorporating the interfacial continuity conditions as constraints through Lagrange's multipliers in the variational formulation. The models developed along this line account for the through-thickness effect, but inevitably increase the unknowns of the system, leading to improved predictions of interlaminar stresses at the expense of increasing computational effort. It should be pointed out that if the results obtained according to a displacement or hybrid model are found unreliable, improvements cannot be made without entire reformulation.

In view of the deficiency of the existing models, it is desirable to have a model which is at the same level of simplicity as the CLT but not at the expense of increasing the DOF. Recently, we have developed an asymptotic theory for anisotropic inhomogeneous and laminated plates on the basis of three-dimensional elasticity without *a priori* assumptions (Tarn and Wang, 1994; Wang and Tarn, 1994). The dominant feature of the theory is that the three-dimensional asymptotic solution for multilayered composite plates can be determined in a consistent and hierarchic way by treating the CLT equations at various levels. There is no need to treat the system layer by layer nor to consider the interfacial continuity conditions in particular. Comparisons of the analytical results with the benchmark solutions show that the asymptotic solution can be very accurate.

Despite the merits of the asymptotic theory, analytic solutions are possible only for very few problems with simple geometries and special laminations. The practical importance of the theory will not be fully realized unless an effective numerical scheme capable of easy and versatile computation for general laminated plates is developed. The present work aims at developing such a computational model based on the asymptotic theory for numerical modeling of multilayered composite plates.

We begin the formulation with the Hellinger–Reissner principle (Washizu, 1982) in which the displacements and transverse stresses are taken to be the functions subject to variation. By means of non-dimensionalization and asymptotic expansion, the H–R functional for the problem is decomposed into a sequence of functionals at various levels. The three-dimensional solution for a problem can be determined hierarchically by using these functionals in conjunction with the finite element method. The solution derived from the leading-order functional provides the first-order approximation. Treatments of the higher-order functionals yield the modifications. As a result of the decomposition and successive integrations, the displacement nodal DOF on the midplane turns out to be the unknowns in the system equations. The total DOF at each level is $3 \times n$ (n denotes the total number of nodes) and the number is *less than* that of a homogeneous Kirchhoff plate in bending and stretching. In the multilevel computations the stiffness matrix remains unchanged; the one generated at the leading-order level is always used at subsequent levels. This is obviously a pleasant feature because reformulation of the stiffness matrix is not necessary once it has been generated. The formulation is 3D in effect yet the modeling only requires 2D discretization on the plane form of the plate for interpolation. There is no need to interpolate in the thickness direction whatsoever. Variations of the displacements and stresses through the thickness are determined analytically. The model is adaptive and hierarchic in that the computation can be carried on to any desired levels using the lower-order approximations, and the higher-order contributions are of diminishing importance.

2. BASIC VARIATIONAL FORMULATION

We consider an anisotropic inhomogeneous plate of uniform thickness $2h$, having at each point one plane of elastic symmetry parallel to the midplane. Let us select a Cartesian coordinate system such that the plane $x_3 = 0$ coincides with the midplane of the plate and the x_3 axis is directed downward from the origin. On the top and bottom faces $x_3 = \pm h$, the transverse loads $q^{\pm}(x_1, x_2)$ are prescribed. Along the edges of the plate, appropriate edge boundary conditions which will be derived in due course are prescribed.

The stress-displacement relations expressed in the geometrical axes are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,3}+u_{3,2} \\ u_{1,3}+u_{3,1} \\ u_{1,2}+u_{2,1} \end{Bmatrix}, \tag{1}$$

where the displacement components are denoted by u_1, u_2 and u_3; the commas denote partial differentiation with respect to the suffix variables. σ_{11}, $\sigma_{22}, \ldots$, and σ_{12} are the stress components. $c_{ij}(i,j = 1, 2, \ldots, 6)$ are the 13 elastic constants of the anisotropic material with one plane of elastic symmetry. The material is considered to be inhomogeneous through the plate thickness, thus, $c_{ij} = c_{ij}(x_3)$. The laminated plate is an important case with piecewisely constant elastic moduli. Obviously, the homogeneous plate is a special case in which c_{ij} are independent of x_3.

The Hellinger–Reissner principle is the basis for constructing the model. The H–R variational functional for the problem takes the form

$$\Pi_R = \int_{-h}^{h}\int_{\Omega}[\tfrac{1}{2}\sigma_{ij}(u_{i,j}+u_{j,i})-B(\sigma_{ij})]\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3-\int_{\Omega^+} q^- u_3\,\mathrm{d}x_1\,\mathrm{d}x_2-\int_{\Omega} q^- u_3\,\mathrm{d}x_1\,\mathrm{d}x_2$$
$$-\int_{-h}^{h}\int_{\Gamma_\sigma}\bar{T}_i u_i\,\mathrm{d}\Gamma\,\mathrm{d}x_3-\int_{-h}^{h}\int_{\Gamma_u}T_i(u_i-\bar{u}_i)\,\mathrm{d}\Gamma\,\mathrm{d}x_3, \quad (2)$$

in which the usual index notation is used. Ω denotes the plate domain on the $x-y$ plane, Ω^+ and Ω^- denote the top face ($x_3 = -h$) and the bottom face ($x_3 = h$) of the plate where the transverse loads $q^{\pm}$ are applied; Γ_σ and Γ_u denote respectively the portion of the edge boundary where tractions $\bar{T}_i$ are prescribed and where displacements $\bar{u}_i$ are prescribed. $B(\sigma_{ij})$ denotes the complementary energy density function such that $\varepsilon_{ij} = \partial B/\partial\sigma_{ij}$.

We shall take the displacements and transverse stresses to be the functions subject to variation and the in-plane stresses to be dependent variables. To this end, let us use (1) to express the strains and σ_{11}, σ_{22}, σ_{12} in terms of u_1, u_2, u_3 and σ_{13}, σ_{23}, σ_{33} as follows:

$$\varepsilon_{11} = \partial B/\partial\sigma_{11} = u_{1,1}, \quad \varepsilon_{22} = \partial B/\partial\sigma_{22} = u_{2,2}, \quad \varepsilon_{12} = \partial B/\partial\sigma_{12} = \tfrac{1}{2}(u_{1,2}+u_{2,1}), \quad (3\text{–}5)$$

$$\varepsilon_{13} = \partial B/\partial\sigma_{13} = \tfrac{1}{2}(s_{45}\sigma_{23}+s_{55}\sigma_{13}), \quad (6)$$

$$\varepsilon_{23} = \partial B/\partial\sigma_{23} = \tfrac{1}{2}(s_{44}\sigma_{23}+s_{45}\sigma_{13}), \quad (7)$$

$$\varepsilon_{33} = \partial B/\partial\sigma_{33} = -[L_{13}\;L_{23}]\begin{Bmatrix}u_1\\u_2\end{Bmatrix}+\sigma_{33}/c_{33}, \quad (8)$$

$$\begin{Bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{12}\end{Bmatrix} = \begin{bmatrix}L_{14} & L_{24}\\ L_{15} & L_{25}\\ L_{16} & L_{26}\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}+\begin{Bmatrix}c_{13}\\c_{23}\\c_{36}\end{Bmatrix}c_{33}^{-1}\sigma_{33}, \quad (9)$$

where

$$L_{13} = \frac{c_{13}}{c_{33}}\frac{\partial}{\partial x_1}+\frac{c_{36}}{c_{33}}\frac{\partial}{\partial x_2}, \quad L_{23} = \frac{c_{36}}{c_{33}}\frac{\partial}{\partial x_1}+\frac{c_{23}}{c_{33}}\frac{\partial}{\partial x_2}, \quad L_{14} = Q_{11}\frac{\partial}{\partial x_1}+Q_{16}\frac{\partial}{\partial x_2},$$
$$L_{24} = Q_{16}\frac{\partial}{\partial x_1}+Q_{12}\frac{\partial}{\partial x_2}, \quad L_{15} = Q_{12}\frac{\partial}{\partial x_1}+Q_{26}\frac{\partial}{\partial x_2}, \quad L_{25} = Q_{26}\frac{\partial}{\partial x_1}+Q_{22}\frac{\partial}{\partial x_2},$$
$$L_{16} = Q_{16}\frac{\partial}{\partial x_1}+Q_{66}\frac{\partial}{\partial x_2}, \quad L_{26} = Q_{66}\frac{\partial}{\partial x_1}+Q_{26}\frac{\partial}{\partial x_2}, \quad Q_{ij} = c_{ij}-c_{i3}c_{j3}/c_{33},$$

$$\begin{bmatrix}s_{55} & s_{45}\\ s_{45} & s_{44}\end{bmatrix} = \begin{bmatrix}c_{55} & c_{45}\\ c_{45} & c_{44}\end{bmatrix}^{-1}.$$

Substituting (3)–(9) into (2) and imposing $\delta\Pi_R = 0$, we obtain

$$\delta\Pi_R = \int_{-h}^{h}\int_{\Omega}[(L_{14}u_1+L_{24}u_2+c_{13}c_{33}^{-1}\sigma_{33})\delta u_{1,1}+(L_{15}u_1+L_{25}u_2+c_{23}c_{33}^{-1}\sigma_{33})\delta u_{2,2}$$
$$-(L_{16}u_1+L_{26}u_2+c_{36}c_{33}^{-1}\sigma_{33})(\delta u_{2,1}+\delta u_{1,2})+\sigma_{33}\delta u_{3,3}+\sigma_{23}(\delta u_{3,2}+\delta u_{2,3})$$
$$+\sigma_{13}(\delta u_{3,1}+\delta u_{1,3})+u_{3,3}\delta\sigma_{33}+(u_{2,3}+u_{3,2})\delta\sigma_{23}+(u_{1,3}+u_{3,1})\delta\sigma_{13}$$
$$+(L_{13}u_1+L_{23}u_2-c_{33}^{-1}\sigma_{33})\delta\sigma_{33}-(s_{44}\sigma_{23}+s_{45}\sigma_{13})\delta\sigma_{23}$$
$$-(s_{45}\sigma_{23}+s_{55}\sigma_{13})\delta\sigma_{13}]\,dx_1\,dx_2\,dx_3-\int_{\Omega^+}q^+\delta u_3\,dx_1\,dx_2-\int_{\Omega^-}q^-\delta u_3\,dx_1\,dx_2$$
$$-\int_{-h}^{h}\int_{\Gamma}T_i\delta u_i\,d\Gamma\,dx_3-\int_{-h}^{h}\int_{\Gamma_u}(u_i-\bar{u}_i)\delta T_i\,d\Gamma\,dx_3 = 0. \tag{10}$$

After integrating by parts the terms associated with $\delta u_{3,3}$, $\delta u_{1,3}$ and $\delta u_{2,3}$, and using Green's theorem, equation (10) becomes

$$\delta\Pi_R = \int_{-h}^{h}\int_{\Omega}\{-(L_{11}u_1+L_{12}u_2+L_{13}\sigma_{33}+\sigma_{13,3})\delta u_1-(L_{12}u_1+L_{22}u_2+L_{23}\sigma_{33}+\sigma_{23,3})\delta u_2$$
$$-(\sigma_{13,1}+\sigma_{23,2}+\sigma_{33,3})\delta u_3+[(u_{2,3}+u_{3,2})-(s_{45}\sigma_{13}+s_{44}\sigma_{23})]\delta\sigma_{23}+[(u_{1,3}+u_{3,1})$$
$$-(s_{55}\sigma_{13}+s_{45}\sigma_{23})]\delta\sigma_{13}+(u_{3,3}+L_{13}u_1+L_{23}u_2-c_{33}^{-1}\sigma_{33})\delta\sigma_{33}\}\,dx_1\,dx_2\,dx_3$$
$$+\oint_{\Gamma}\int_{-h}^{h}[(\sigma_{11}n_1+\sigma_{12}n_2-T_1)\delta u_1+(\sigma_{12}n_1+\sigma_{22}n_2-T_2)\delta u_2$$
$$+(\sigma_{13}n_1+\sigma_{23}n_2-T_3)\delta u_3]\,dx_3\,d\Gamma+\int_{\Omega^+}\sigma_{13}^{+}\delta u_1\,dx_1\,dx_2$$
$$-\int_{\Omega^-}\sigma_{13}^{-}\delta u_1\,dx_1\,dx_2+\int_{\Omega^+}\sigma_{23}^{+}\delta u_2\,dx_1\,dx_2-\int_{\Omega^-}\sigma_{23}^{-}\delta u_2\,dx_1\,dx_2$$
$$+\int_{\Omega^+}(\sigma_{33}-q^+)\delta u_3\,dx_1\,dx_2-\int_{\Omega^-}(\sigma_{33}+q^-)\delta u_3\,dx_1\,dx_2-\int_{-h}^{h}\int_{\Gamma_u}(u_i-\bar{u}_i)\delta T_i\,d\Gamma\,dx_3 = 0, \tag{11}$$

where

$$L_{11} = \left(Q_{11}\frac{\partial^2}{\partial x_1^2}+2Q_{16}\frac{\partial^2}{\partial x_1\,\partial x_2}+Q_{66}\frac{\partial^2}{\partial x_2^2}\right),$$
$$L_{12} = \left(Q_{16}\frac{\partial^2}{\partial x_1^2}+(Q_{12}+Q_{66})\frac{\partial^2}{\partial x_1\,\partial x_2}+Q_{26}\frac{\partial^2}{\partial x_2^2}\right),$$
$$L_{22} = \left(Q_{66}\frac{\partial^2}{\partial x_1^2}+2Q_{26}\frac{\partial^2}{\partial x_1\,\partial x_2}+Q_{22}\frac{\partial^2}{\partial x_2^2}\right).$$

The Euler-Lagrange equations derived from (11) are precisely those obtained by direct elimination of the three-dimensional equations (Wang and Tarn, 1994). Therefore, the variational functional (11) is a weak formulation which involves only first derivatives of the field variables.

The admissible boundary conditions for the problem can be derived from the boundary integral terms in (11). The lateral boundary conditions are those usually prescribed on the top and bottom surfaces of the plate:

$$
\begin{aligned}
&[\sigma_{13}\ \sigma_{23}] = [0,0] && \text{on} \quad x_3 = \pm h, \\
&\sigma_{33} = -q^{-}(x_1, x_2) && \text{on} \quad x_3 = -h, \\
&\sigma_{33} = q^{+}(x_1, x_2) && \text{on} \quad x_3 = h,
\end{aligned} \tag{12}
$$

where the transverse loads $q^{\pm}$ are positive downwards in accordance with the positive direction of the x_3 coordinate.

The admissible edge conditions require one member of each pair of the following quantities be satisfied along the edges:

$$
\begin{aligned}
&\sigma_{11}n_1 + \sigma_{12}n_2 = \bar{T}_1 && \text{or} \quad u_1 = \bar{u}_1, \\
&\sigma_{12}n_1 + \sigma_{22}n_2 = \bar{T}_2 && \text{or} \quad u_2 = \bar{u}_2, \\
&\sigma_{13}n_1 + \sigma_{23}n_2 = \bar{T}_3 && \text{or} \quad u_3 = \bar{u}_3.
\end{aligned} \tag{13}
$$

The above edge conditions are nothing but the traction or displacement boundary conditions for the plate in three-dimensional elasticity; however, they are not easy to satisfy exactly point by point on the edge surfaces. Under these circumstances, one must resort to Saint-Venant's principle to replace the edge conditions by equivalent ones. This is common to almost all plate theories. The solution thus obtained is considered to be valid everywhere except at the edge boundary layers. The applicability of Saint-Venant's principle to homogeneous plates has been under investigation (see, e.g. Gregory and Wan, 1985), but little has been done for laminated plates. We shall not investigate the boundary layer effect in this paper but focus rather on determining the interior solution.

3. NON-DIMENSIONALIZATION AND ASYMPTOTIC EXPANSION

As in the asymptotic theory (Wang and Tarn, 1994), let us make the variational functional (11) dimensionless by introducing

$$
\begin{aligned}
&x = x_1/l, \qquad y = x_2/l, \qquad z = x_3/h, \\
&u_1 = \varepsilon l u, \qquad u_2 = \varepsilon l v, \qquad u_3 = l w, \\
&\sigma_{11} = \varepsilon Q \sigma_x, \qquad \sigma_{22} = \varepsilon Q \sigma_y, \qquad \sigma_{12} = \varepsilon Q \sigma_{xy}, \\
&\sigma_{13} = \varepsilon^2 Q \sigma_{xz}, \qquad \sigma_{23} = \varepsilon^2 Q \sigma_{yz}, \qquad \sigma_{33} = \varepsilon^3 Q \sigma_z,
\end{aligned} \tag{14}
$$

where $\varepsilon = h/l$ is a dimensionless parameter, usually much less than 1; l denotes a typical in-plane dimension of the plate, and $-1 \leqslant z \leqslant 1$; Q represents a reference elastic modulus.

Introducing (14) and the displacement boundary condition in (10), we have

$$
\begin{aligned}
\delta\Pi_R = c_{33}h^3 \Bigg[\int_{-1}^{1} \int_{\Omega(x,y)} & \{[(\mathbf{L}_1\mathbf{u})^{\mathrm{T}} + \varepsilon^2 \sigma_z \mathbf{L}_2^{\mathrm{T}}]\mathbf{D}_1 \delta\mathbf{u} + \boldsymbol{\sigma}_S^{\mathrm{T}}(\mathbf{D}\delta w + \delta\mathbf{u}_{,z}) + \sigma_z \delta w_{,z} \\
& + (w_{,z} + \varepsilon^2 \mathbf{L}_3 \mathbf{u} - \varepsilon^4 \tilde{c}_{33}^{-1} \sigma_z)\delta\sigma_z + (\mathbf{D}^{\mathrm{T}} w + \mathbf{u}_{,z}^{\mathrm{T}} - \varepsilon^2 \boldsymbol{\sigma}_S^{\mathrm{T}} \mathbf{S})\delta\boldsymbol{\sigma}_S\} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
& - \int_{\Omega^+} \tilde{q}^{+} \delta w \, \mathrm{d}x \, \mathrm{d}y - \int_{\Omega^-} \tilde{q}^{-} \delta w \, \mathrm{d}x \, \mathrm{d}y - \int_{-1}^{1} \int_{\Gamma} (\mathbf{P}^{\mathrm{T}} \delta\mathbf{u} + p_3 \delta w) \, \mathrm{d}\bar{\Gamma} \, \mathrm{d}z \Bigg] = 0,
\end{aligned} \tag{15}
$$

where the superscript $^{\mathrm{T}}$ stands for the transpose, $\mathrm{d}\bar{\Gamma}$ is the dimensionless arc length along the edges, p_3 and $\tilde{q}^{\pm}$ are the dimensionless loads, and

$$\mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \boldsymbol{\sigma}_S = \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix}, \quad \mathbf{D} = \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix},$$

$$\mathbf{L}_1 = \begin{bmatrix} d_{14} & d_{24} \\ d_{15} & d_{25} \\ d_{16} & d_{26} \end{bmatrix}, \quad \mathbf{L}_2 = c_{33}^{-1} \begin{Bmatrix} c_{13} \\ c_{23} \\ c_{36} \end{Bmatrix}, \quad \mathbf{L}_3 = [d_{13} \quad d_{23}], \quad \mathbf{S} = \begin{bmatrix} \tilde{s}_{55} & \tilde{s}_{45} \\ \tilde{s}_{45} & \tilde{s}_{44} \end{bmatrix},$$

$$d_{13} = \tilde{c}_{13}\frac{\partial}{\partial x} + \tilde{c}_{36}\frac{\partial}{\partial y}, \quad d_{23} = \tilde{c}_{36}\frac{\partial}{\partial x} + \tilde{c}_{23}\frac{\partial}{\partial y}, \quad d_{14} = \tilde{Q}_{11}\frac{\partial}{\partial x} + \tilde{Q}_{16}\frac{\partial}{\partial y},$$

$$d_{24} = \tilde{Q}_{16}\frac{\partial}{\partial x} + \tilde{Q}_{12}\frac{\partial}{\partial y}, \quad d_{15} = \tilde{Q}_{12}\frac{\partial}{\partial x} + \tilde{Q}_{26}\frac{\partial}{\partial y}, \quad d_{25} = \tilde{Q}_{26}\frac{\partial}{\partial x} + \tilde{Q}_{22}\frac{\partial}{\partial y},$$

$$d_{16} = \tilde{Q}_{16}\frac{\partial}{\partial x} + \tilde{Q}_{66}\frac{\partial}{\partial y}, \quad d_{26} = \tilde{Q}_{66}\frac{\partial}{\partial x} + \tilde{Q}_{26}\frac{\partial}{\partial y}, \quad \tilde{Q}_{ij} = Q_{ij}/Q, \quad \tilde{s}_{ij} = Qs_{ij},$$

$$\tilde{c}_{ij} = c_{ij}/Q, \quad \mathbf{P}^{\mathrm{T}} = [T_1/\varepsilon Q \quad T_2/\varepsilon Q], \quad p_3 = T_3/\varepsilon^2 Q, \quad \tilde{q}^{\pm} = q^{\pm}/\varepsilon^3 Q.$$

Integrating by parts the terms associated with $\delta\mathbf{u}_{,z}$ and $\delta w_{,z}$ in (15), and using the lateral boundary conditions (12), we obtain

$$\begin{aligned}\delta\Pi_R = c_{33}h^3\Bigg[\int_{-1}^{1}\int_{\Omega} \{[(\mathbf{L}_1\mathbf{u})^{\mathrm{T}} + \varepsilon^2\sigma_z\mathbf{L}_2^{\mathrm{T}}]\mathbf{D}_1\delta\mathbf{u} - \boldsymbol{\sigma}_{S,z}^{\mathrm{T}}\delta\mathbf{u} + \boldsymbol{\sigma}_S^{\mathrm{T}}\mathbf{D}\delta w - \sigma_{z,z}\delta w \\ + (w_{,z} + \varepsilon^2\mathbf{L}_3\mathbf{u} - \varepsilon^4\tilde{c}_{33}^{-1}\sigma_z)\delta\sigma_z + (\mathbf{D}^{\mathrm{T}}w + \mathbf{u}_{,z}^{\mathrm{T}} - \varepsilon^2\boldsymbol{\sigma}_S^{\mathrm{T}}\mathbf{S})\delta\boldsymbol{\sigma}_S\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ - \int_{-1}^{1}\int_{\Gamma} (\mathbf{P}^{\mathrm{T}}\delta\mathbf{u} + p_3\delta w)\,\mathrm{d}\bar{\Gamma}\,\mathrm{d}z\Bigg] = 0. \quad (16)\end{aligned}$$

We now expand the displacements and the transverse stresses in powers of ε^2 as given by

$$f(x,y,z;\varepsilon) = f_{(0)}(x,y,z) + \varepsilon^2 f_{(1)}(x,y,z) + \varepsilon^4 f_{(2)}(x,y,z) + \ldots \quad (17)$$

After substituting (17) in (16) and collecting like-power terms of ε, we obtain

$$\delta\Pi_R = c_{33}h^3(\delta\Pi_{(0)} + \varepsilon^2\delta\Pi_{(1)} + \varepsilon^4\delta\Pi_{(2)} + \ldots) = 0, \quad (18)$$

in which

$$\begin{aligned}\delta\Pi_{(0)} = \int_{-1}^{1}\int_{\Omega} [(\mathbf{L}_1\mathbf{u}_{(0)})^{\mathrm{T}}\mathbf{D}_1\delta\mathbf{u} - \boldsymbol{\sigma}_{S(0),z}^{\mathrm{T}}\delta\mathbf{u} + \boldsymbol{\sigma}_{S(0)}^{\mathrm{T}}\mathrm{D}\delta w - \sigma_{z(0),z}\delta w + w_{(0),z}\delta\sigma_z \\ + (\mathbf{D}^{\mathrm{T}}w_{(0)} + \mathbf{u}_{(0),z}^{\mathrm{T}})\delta\boldsymbol{\sigma}_S]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \int_{-1}^{1}\int_{\Gamma} (\mathbf{P}_{(0)}^{\mathrm{T}}\delta\mathbf{u} + p_{3(0)}\delta w)\,\mathrm{d}\bar{\Gamma}\,\mathrm{d}z, \quad (19)\end{aligned}$$

$$\delta\Pi_{(1)} = \int_{-1}^{1}\int_{\Omega}[(\mathbf{L}_1\mathbf{u}_{(1)})^{\mathrm{T}}\mathbf{D}_1\delta\mathbf{u} - \boldsymbol{\sigma}_{S(1),z}^{\mathrm{T}}\delta\mathbf{u} + \boldsymbol{\sigma}_{S(1)}^{\mathrm{T}}\mathbf{D}\delta w - \sigma_{z(1),z}\delta w + w_{(1),z}\delta\sigma_z$$

$$+(\mathbf{D}^{\mathrm{T}}w_{(1)} + \mathbf{u}_{(1),z}^{\mathrm{T}})\delta\boldsymbol{\sigma}_S]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{-1}^{1}\int_{\Omega}[\sigma_{z(0)}\mathbf{L}_2^{\mathrm{T}}\mathbf{D}_1\delta\mathbf{u} + \mathbf{L}_3\mathbf{u}_{(0)}\delta\sigma_z$$

$$-\boldsymbol{\sigma}_{S(0)}^{\mathrm{T}}\mathbf{S}\delta\boldsymbol{\sigma}_S]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \int_{-1}^{1}\int_{\Gamma_u}(\mathbf{P}_{(1)}^{\mathrm{T}}\delta\mathbf{u} + p_{3(1)}\delta w)\,\mathrm{d}\bar{\Gamma}\,\mathrm{d}z, \quad (20)$$

$$\delta\Pi_{(k)} = \int_{-1}^{1}\int_{\Omega}[(\mathbf{L}_1\mathbf{u}_{(k)})^{\mathrm{T}}\mathbf{D}_1\delta\mathbf{u} - \boldsymbol{\sigma}_{S(k),z}^{\mathrm{T}}\delta\mathbf{u} + \boldsymbol{\sigma}_{S(k)}^{\mathrm{T}}\mathbf{D}\delta w - \sigma_{z(k),z}\delta w + w_{(k),z}\delta\sigma_z$$

$$+(\mathbf{D}^{\mathrm{T}}w_{(k)} + \mathbf{u}_{(k),z}^{\mathrm{T}})\delta\boldsymbol{\sigma}_S]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{-1}^{1}\int_{\Omega}[\sigma_{z(k-1)}\mathbf{L}_2^{\mathrm{T}}\mathbf{D}_1\delta\mathbf{u} + (\mathbf{L}_3\mathbf{u}_{(k-1)} - \tilde{c}_{33}^{-1}\sigma_{z(k-2)})\delta\sigma_z$$

$$-\boldsymbol{\sigma}_{S(k-1)}^{\mathrm{T}}\mathbf{S}\delta\boldsymbol{\sigma}_S]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \int_{-1}^{1}\int_{\Gamma_u}(\mathbf{P}_{(k)}^{\mathrm{T}}\delta\mathbf{u} + p_{3(k)}\delta w)\,\mathrm{d}\bar{\Gamma}\,\mathrm{d}z, \quad (k = 2, 3, \ldots) \quad (21)$$

in which $\mathbf{P}_{(k)}$ and $p_{3(k)}$ denote the edge tractions at the kth level.

Equation (18) is an expression of $\delta\Pi_R$ in terms of even powers of ε. Thus, we have decomposed the H–R functional into functionals of various orders; $\delta\Pi_{(0)} = 0$ represents the leading-order approximation to the stationary condition $\delta\Pi_R = 0$, $\delta\Pi_{(1)} = 0$ provides the ε^2-order modification, $\delta\Pi_{(2)} = 0$ the ε^4-order modification, and so on.

The dimensionless boundary conditions associated with each level are as follows.

At the leading-order:

$$\boldsymbol{\sigma}_{S(0)} = 0 \quad \text{on} \quad z = \pm 1, \quad (22)$$

$$\sigma_{z(0)} = -\tilde{q}^-(x, y) \quad \text{on} \quad z = -1, \quad (23)$$

$$\sigma_{z(0)} = \tilde{q}^+(x, y) \quad \text{on} \quad z = 1, \quad (24)$$

and along the edges,

$$\sigma_{x(0)}n_x + \sigma_{xy(0)}n_y = \bar{T}_1 \quad \text{or} \quad u_{(0)} = \bar{u}, \quad (25)$$

$$\sigma_{xy(0)}n_x + \sigma_{y(0)}n_y = \bar{T}_2 \quad \text{or} \quad v_{(0)} = \bar{v}, \quad (26)$$

$$\sigma_{xz(0)}n_x + \sigma_{yz(0)}n_y = \bar{T}_3 \quad \text{or} \quad w_{(0)} = \bar{w}. \quad (27)$$

At the order $\varepsilon^{2k}(k = 1, 2, \ldots)$:

$$\boldsymbol{\sigma}_{S(k)} = 0 \quad \text{on} \quad z = \pm 1, \quad (28)$$

$$\sigma_{z(k)} = 0 \quad \text{on} \quad z = \pm 1, \quad (29)$$

and along the edges,

$$\sigma_{x(k)}n_x + \sigma_{xy(k)}n_y = 0 \quad \text{or} \quad u_{(k)} = 0, \quad (30)$$

$$\sigma_{xy(k)}n_x + \sigma_{y(k)}n_y = 0 \quad \text{or} \quad v_{(k)} = 0, \quad (31)$$

$$\sigma_{xz(k)}n_x + \sigma_{yz(k)}n_y = 0 \quad \text{or} \quad w_{(k)} = 0. \quad (32)$$

Note that the second integrals in the expressions of $\delta\Pi_{(1)}$ and $\delta\Pi_{(k)}$ are determined using the lower-order solutions. They play the same role as the second integral in $\delta\Pi_{(0)}$, leading to a generalized force vector in the system equations at the corresponding level. Furthermore, the first integral in (20) and in (21) are essentially of the same form as the corresponding one in (19). In consequence the stiffness matrix is expected to remain unchanged; the one generated at the leading-order level is always used at subsequent levels. This will become evident as we proceed.

4. ASYMPTOTIC FINITE ELEMENT FORMULATION

The asymptotic variational formulation provides a basis for constructing a computational model in conjunction with the finite element method. In the finite element formulation the x–y plane of the plate domain is discretized into a finite element assemblage and appropriate interpolation functions are used to represent the variations of the field variables within the elements. Since the variational functionals for the problem involve the transverse stresses and first derivatives of the displacements as functions of x and y, the admissible interpolation functions for the transverse stresses require C^0 continuity whereas those for the displacements should be continuous in each element and conforming on inter-element boundaries.

4.1. *The leading-order level*

Let us consider the leading-order level first. The variations of the field variables $\mathbf{u}$, w, $\boldsymbol{\sigma}_S$ and σ_z within an element are expressed in terms of the generalized nodal DOF as

$$\mathbf{u}_{(0)}(x,y,z) = \mathbf{N}\mathbf{u}^{e}_{(0)}(z), \tag{33}$$

$$w_{(0)}(x,y,z) = \mathbf{N}_w\mathbf{w}^{e}_{(0)}(z), \tag{34}$$

$$\boldsymbol{\sigma}_{S(0)}(x,y,z) = \mathbf{N}_S\boldsymbol{\sigma}^{e}_{S(0)}(z), \tag{35}$$

$$\sigma_{z(0)}(x,y,z) = \mathbf{N}_z\boldsymbol{\sigma}^{e}_{z(0)}(z), \tag{36}$$

in which $\mathbf{N}$, $\mathbf{N}_w$, $\mathbf{N}_S$ and $\mathbf{N}_z$ denote the interpolation functions of x and y but not of z, whereas the generalized nodal DOF $\mathbf{u}^{e}_{(0)} = [u_{(0)}v_{(0)}]^{\mathrm{T}}$, $\mathbf{w}^{e}_{(0)}$, $\boldsymbol{\sigma}^{e}_{S(0)} = [\sigma_{xz(0)}\sigma_{yz(0)}]^{\mathrm{T}}$ and $\sigma^{e}_{z(0)}$ are functions of z but not of x and y. The displacements and the transverse stresses may be interpolated independently. It appears at this point that there are six generalized DOF at a node, however, through successive integration the transverse stress DOF can be eliminated in the element level, only the midplane displacement DOF enter the final system equations at each level. The dependence of the generalized nodal DOF upon z can be determined analytically.

Substitution of (33)–(36) into the expression $\delta\Pi_{(0)}$ leads to

$$\begin{aligned}\delta\Pi_{(0)} = \sum_{e}\Bigg[\int_{-1}^{1}\Bigg\{&\left[(\mathbf{u}^{e}_{(0)})^{\mathrm{T}}\Psi^{e}-\left(\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{\sigma}^{e}_{S(0)}\right)^{\mathrm{T}}\Phi^{e}_{1}\right]\delta\mathbf{u}+\left[(\boldsymbol{\sigma}^{e}_{S(0)})^{\mathrm{T}}\Phi^{e}_{2}-\left(\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{\sigma}^{e}_{z(0)}\right)^{\mathrm{T}}\Phi^{e}_{3}\right]\delta\mathbf{w}\\ &+\left(\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{w}^{e}_{(0)}\right)^{\mathrm{T}}(\Phi^{e}_{3})^{\mathrm{T}}\delta\boldsymbol{\sigma}_{z}+\left[(\mathbf{w}^{n}_{(0)})^{\mathrm{T}}(\Phi^{e}_{2})^{\mathrm{T}}+\left(\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{u}^{e}_{(0)}\right)^{\mathrm{T}}(\Phi^{e}_{1})^{\mathrm{T}}\right]\delta\boldsymbol{\sigma}_{S}\Bigg\}\,\mathrm{d}z\\ &-\int_{-1}^{1}\{(\mathbf{P}^{e}_{(0)})^{\mathrm{T}}\delta\mathbf{u}+(\mathbf{P}^{e}_{w(0)})^{\mathrm{T}}\delta\mathbf{w}\}\,\mathrm{d}z\Bigg],\end{aligned} \tag{37}$$

where

$$\Psi^e = (\Psi^e)^{\mathrm{T}} = \int_{\Omega_e} (\mathbf{L}_1\mathbf{N})^{\mathrm{T}}(\mathbf{D}_1\mathbf{N})\,\mathrm{d}x\,\mathrm{d}y, \quad \Phi_1^e = \int_{\Omega_e} \mathbf{N}_S^{\mathrm{T}}\mathbf{N}\,\mathrm{d}x\,\mathrm{d}y,$$

$$\Phi_2^e = \int_{\Omega_e} \mathbf{N}_S^{\mathrm{T}}(\mathbf{D}\mathbf{N}_w)\,\mathrm{d}x\,\mathrm{d}y, \qquad \Phi_3^e = \int_{\Omega_e} \mathbf{N}_z^{\mathrm{T}}\mathbf{N}_w\,\mathrm{d}x\,\mathrm{d}y,$$

$$\mathbf{P}_{(0)}^e = \int_{\Gamma_\sigma^e} \mathbf{N}^{\mathrm{T}}\mathbf{T}_u\,\mathrm{d}\bar{\Gamma}, \qquad \mathbf{P}_{w(0)}^e = \int_{\Gamma_\sigma^e} \mathbf{N}_w^{\mathrm{T}}\mathbf{T}_w\,\mathrm{d}\bar{\Gamma},$$

in which Ω_e denotes the plane area of an element. The tractions on the element boundaries that coincide with the edge boundary with prescribed tractions are given by $\mathbf{T}_u = [\bar{T}_1\ \bar{T}_2]$, $\mathbf{T}_w = \bar{T}_3$. If the same interpolation functions are used for the displacements and for the transverse stresses, we have $\Phi_1^e = (\Phi_1^e)^{\mathrm{T}}$ and $\Phi_3^e = (\Phi_3^e)^{\mathrm{T}}$.

After assembling the element matrices into global matrices, $\delta\Pi_{(0)} = 0$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{w}_{(0)}^n = \mathbf{0}, \tag{38}$$

$$\Phi_1\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{u}_{(0)}^n + \Phi_2\mathbf{w}_{(0)}^n = \mathbf{0}, \tag{39}$$

$$\Phi_1^{\mathrm{T}}\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{\sigma}_{S(0)}^n - \Psi\mathbf{u}_{(0)}^n + \mathbf{P}_{(0)} = \mathbf{0}, \tag{40}$$

$$\Phi_3^{\mathrm{T}}\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{\sigma}_{z(0)}^n - \Phi_2^{\mathrm{T}}\boldsymbol{\sigma}_{S(0)}^n + \mathbf{P}_{w(0)} = \mathbf{0}, \tag{41}$$

where $\mathbf{u}_{(0)}^n$, $\mathbf{w}_{(0)}^n$, $\boldsymbol{\sigma}_{S(0)}^n$ and $\boldsymbol{\sigma}_{z(0)}^n$ consist of all the nodal DOF in the domain. The superscript e has been dropped to indicate the system matrices.

Equations (38)–(41) can be integrated with respect to z in succession. As a result of the successive integration, we have

$$\mathbf{w}_{(0)}^n(z) = \mathbf{w}_0^n = \text{constant}, \tag{42}$$

$$\mathbf{u}_{(0)}^n(z) = \mathbf{u}_0^n - z\Phi_1^{-1}\Phi_2\mathbf{w}_0^n, \tag{43}$$

$$\boldsymbol{\sigma}_{S(0)}^n(z) = (\Phi_1^{\mathrm{T}})^{-1}\int_{-1}^{z}[\Psi(\mathbf{u}_0^n - \eta\Phi_1^{-1}\Phi_2\mathbf{w}_0^n) - \mathbf{P}_{(0)}]\,\mathrm{d}\boldsymbol{\eta}, \tag{44}$$

$$\begin{aligned}\boldsymbol{\sigma}_{z(0)}^n(z) &= (\Phi_3^{\mathrm{T}})^{-1}\left\{\Phi_2^{\mathrm{T}}(\Phi_1^{\mathrm{T}})^{-1}\int_{-1}^{z}\int_{-1}^{z}[\Psi(\mathbf{u}_0^n - z\Phi_1^{-1}\Phi_2\mathbf{w}_0^n) - \mathbf{P}_{(0)}]\,\mathrm{d}z\,\mathrm{d}z - \int_{-1}^{z}\mathbf{P}_{w(0)}\,\mathrm{d}z\right\} - \tilde{\mathbf{q}}_n^- \\ &= (\Phi_3^{\mathrm{T}})^{-1}\left\{\Phi_2^{\mathrm{T}}(\Phi_1^{\mathrm{T}})^{-1}\int_{-1}^{z}(z-\eta)[\Psi(\mathbf{u}_0^n - \eta\Phi_1^{-1}\Phi_2\mathbf{w}_0^n) - \mathbf{P}_{(0)}]\,\mathrm{d}\eta - \int_{-1}^{z}\mathbf{P}_{w(0)}\,\mathrm{d}\eta\right\} - \tilde{\mathbf{q}}_n^-,\end{aligned} \tag{45}$$

where $\mathbf{u}_0^n$ and $\mathbf{w}_0^n$ are the yet unknown integration constants. Obviously, they represent the leading-order displacements on the midplane $z = 0$. $\tilde{\mathbf{q}}_n^-$ consists of the nodal values of the lateral load acting on $z = -1$. The double integral in (45) is reduced to the single integral using integration by parts. The dependence of the integrands upon the thickness coordinate has been changed from z to η.

Consideration of the boundary conditions is in order. The associated lateral boundary conditions on $z = -1$ are satisfied by (44) and (45). The ones on $z = 1$ are satisfied by letting

$$\int_{-1}^{1} \Psi(\mathbf{u}_0^n - z\Phi_1^{-1}\Phi_2\mathbf{w}_0^n)\,dz = \int_{-1}^{1} \mathbf{P}_{(0)}\,dz, \tag{46}$$

$$(\Phi_3^{\mathrm{T}})^{-1}\left\{\Phi_2^{\mathrm{T}}(\Phi_1^{\mathrm{T}})^{-1}\int_{-1}^{1}(1-\eta)[\Psi(\mathbf{u}_0^n-\eta\Phi_1^{-1}\Phi_2\mathbf{w}_0^n)-\mathbf{P}_{(0)}]\,d\eta-\int_{-1}^{1}\mathbf{P}_{w(0)}\,d\eta\right\}=\tilde{\mathbf{q}}_n^- + \tilde{\mathbf{q}}_n^+. \tag{47}$$

These equations can be combined and written in the matrix form as

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{uw} \\ \mathbf{K}_{wu} & \mathbf{K}_{ww} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0^n \\ \mathbf{w}_0^n \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_{u0} \\ \mathbf{F}_{w0} \end{Bmatrix}, \tag{48}$$

where

$$\mathbf{K}_{uu} = \int_{-1}^{1}\Psi\,dz,\quad \mathbf{K}_{uw} = -\left(\int_{-1}^{1} z\Psi\,dz\right)\Phi_1^{-1}\Phi_2,\quad \mathbf{K}_{wu} = -\Phi_2^{\mathrm{T}}(\Phi_1^{\mathrm{T}})^{-1}\left(\int_{-1}^{1} z\Psi\,dz\right),$$

$$\mathbf{K}_{ww} = \Phi_2^{\mathrm{T}}(\Phi_1^{\mathrm{T}})^{-1}\left(\int_{-1}^{1} z^2\Psi\,dz\right)\Phi_1^{-1}\Phi_2,\quad \mathbf{F}_{w0} = \Phi_3^{\mathrm{T}}(\tilde{\mathbf{q}}^- + \tilde{\mathbf{q}}^+) + \mathbf{V}_0 - \Phi_2^{\mathrm{T}}(\Phi_1^{\mathrm{T}})^{-1}\mathbf{M}_0,$$

$$\mathbf{F}_{u0} = \int_{-1}^{1}\mathbf{P}_{(0)}\,dz,\quad \mathbf{V}_0 = \int_{-1}^{1}\mathbf{P}_{w(0)}\,dz,\quad \mathbf{M}_0 = \int_{-1}^{1} z\mathbf{P}_{(0)}\,dz,$$

$\mathbf{F}_{u0}$, $\mathbf{V}_0$ and $\mathbf{M}_0$ represent the resultant in-plane force, the shear force and moment across the thickness.

Equation (48) consists of $3\times n$ algebraic equations for $3\times n$ nodal DOF of a domain with n nodes. The stiffness matrix is symmetric because $\mathbf{K}_{uu}$ and $\mathbf{K}_{ww}$ are symmetric and $\mathbf{K}_{wu} = \mathbf{K}_{uw}^{\mathrm{T}}$. Solution to (48) must be supplemented with appropriate edge conditions. Once the nodal values u_{0i}, v_{0i} and w_{0i} have been determined, the leading-order displacements and transverse stresses are obtained using (42)–(45). The in-plane stresses are determined using the dimensionless form of (9):

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} d_{14} & d_{24} \\ d_{15} & d_{25} \\ d_{16} & d_{26} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} + \varepsilon^2 \begin{Bmatrix} \tilde{c}_{13} \\ \tilde{c}_{23} \\ \tilde{c}_{36} \end{Bmatrix} \sigma_z. \tag{49}$$

It follows that the leading-order in-plane stresses are given by

$$\begin{Bmatrix} \sigma_{x(0)} \\ \sigma_{y(0)} \\ \sigma_{xy(0)} \end{Bmatrix} = \begin{bmatrix} d_{14} & d_{24} \\ d_{15} & d_{25} \\ d_{16} & d_{26} \end{bmatrix} \begin{Bmatrix} u_{(0)} \\ v_{(0)} \end{Bmatrix}. \tag{50}$$

To use the edge conditions (25)–(27) along with (48) properly calls for attention. In case tractions are prescribed along the edges, the displacement DOF at the edge nodes are unknown and the generalized force vectors are obtained simply by substituting the edge loads into the expressions for $\mathbf{P}_{(0)}$, $\mathbf{P}_{w(0)}$, $\mathbf{V}_0$ and $\mathbf{M}_0$. The situation is a little more involved in case displacements are prescribed. The midplane displacements must be determined using (33) and (34) in conjunction with (42) and (43) before solving (48) for the generalized force vector at the edge nodes. Here an inconsistency may occur; the displacements given by (43) may not be compatible with the prescribed edge displacements unless the prescribed ones are linear in z. Under this circumstance, one relies on Saint-Venant's principle to disregard the boundary layer effect, keeping in mind that the solution thus obtained represents an interior solution for the problem. For a mixed-type edge condition the treatment is similar; either one of the components of $\mathbf{u}_0^n$; $\mathbf{F}_{u0}$ and $\mathbf{w}_0^n$; $\mathbf{F}_{w0}$ at the edge nodes must be determined before solving (48).

4.2. *Higher-order modifications*

The asymptotic formulation can be carried on to the ε^2 order. The field variables are interpolated according to (27)–(30), then they are substituted into $\delta\Pi_{(1)}$ to give

$$\delta\Pi_{(1)} = \int_{-1}^{1} \left\{ \left[(\mathbf{u}_{(1)}^n)^{\mathrm{T}}\Psi - \left(\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{\sigma}_{S(1)}^n\right)^{\mathrm{T}}\Phi_1 + (\boldsymbol{\sigma}_{z(0)}^n)^{\mathrm{T}}\mathbf{f} \right]\delta\mathbf{u} + \left[(\boldsymbol{\sigma}_{S(1)}^n)^{\mathrm{T}}\Phi_2 - \left(\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{\sigma}_{z(1)}^n\right)^{\mathrm{T}}\Phi_3 \right]\delta\mathbf{w} \right.$$

$$\left. + \left[\left(\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{w}_{(1)}^n\right)^{\mathrm{T}}\Phi_3^{\mathrm{T}} + (\mathbf{u}_{(0)}^n)^{\mathrm{T}}\mathbf{f}^{\mathrm{T}} \right]\delta\boldsymbol{\sigma}_z + \left[(\mathbf{w}_{(1)}^n)^{\mathrm{T}}\Phi_2^{\mathrm{T}} + \left(\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{u}_{(1)}^n\right)^{\mathrm{T}}\Phi_1^{\mathrm{T}} - (\boldsymbol{\sigma}_{S(0)}^n)^{\mathrm{T}}\mathbf{g} \right]\delta\boldsymbol{\sigma}_S \right\} \mathrm{d}z$$

$$- \int_{-1}^{1} \{ (\mathbf{P}_{(1)}^e)^{\mathrm{T}}\delta\mathbf{u} + (\mathbf{P}_{w(1)}^e)^{\mathrm{T}}\delta\mathbf{w} \}\,\mathrm{d}z, \quad (51)$$

where $\mathbf{f}$, $\mathbf{g}$ are obtained by assembling the element matrices:

$$\mathbf{f}^e = \int_{\Omega_e} \mathbf{N}_z^{\mathrm{T}}(\mathbf{L}_2^{\mathrm{T}}\mathbf{D}_1\mathbf{N})\,\mathrm{d}x\,\mathrm{d}y, \quad \mathrm{g}^e = \int_{\Omega_e} \mathbf{N}_S^{\mathrm{T}}\mathbf{S}\mathbf{N}_S\,\mathrm{d}x\,\mathrm{d}y.$$

The condition $\delta\Pi_{(1)} = 0$ gives

$$\Phi_3 \frac{\mathrm{d}}{\mathrm{d}z}\mathbf{w}_{(1)}^n + \mathbf{f}\mathbf{u}_{(0)}^n = \mathbf{0}, \quad (52)$$

$$\Phi_1 \frac{\mathrm{d}}{\mathrm{d}z}\mathbf{u}_{(1)}^n + \Phi_2\mathbf{w}_{(1)}^n - \mathbf{g}\boldsymbol{\sigma}_{S(0)}^n = \mathbf{0}, \quad (53)$$

$$\Phi_1^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{\sigma}_{S(1)}^n - \Psi\mathbf{u}_{(1)}^n - \mathbf{f}^{\mathrm{T}}\boldsymbol{\sigma}_{z(0)}^n - \mathbf{P}_{(1)} = \mathbf{0}, \quad (54)$$

$$\Phi_3^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{\sigma}_{z(1)}^n - \Phi_2^{\mathrm{T}}\boldsymbol{\sigma}_{S(1)}^n - \mathbf{P}_{w(1)} = \mathbf{0}. \quad (55)$$

Successive integration of (52)–(55) yields

$$\mathbf{w}_{(1)}^n(z) = \mathbf{w}_1^n + \mathbf{G}(z), \quad (56)$$

$$\mathbf{u}_{(1)}^n(z) = \mathbf{u}_1^n - z\Phi_1^{-1}\Phi_2\mathbf{w}_1^n + \mathbf{H}(z), \quad (57)$$

$$\boldsymbol{\sigma}_{S(1)}^n(z) = (\Phi_1^{\mathrm{T}})^{-1} \int_{-1}^{z} \Psi(\mathbf{u}_1^n - \eta\Phi_1^{-1}\Phi_2\mathbf{w}_1^n - \mathbf{P}_{(1)})\,\mathrm{d}\eta + \mathbf{I}(z), \quad (58)$$

$$\boldsymbol{\sigma}_{z(1)}^n(z) = (\Phi_3^{\mathrm{T}})^{-1} \left\{ \Phi_2^{\mathrm{T}}(\Phi_1^{\mathrm{T}})^{-1} \int_{-1}^{z} (z-\eta)\Psi(\mathbf{u}_1^n - \eta\Phi_1^{-1}\Phi_2\mathbf{w}_1^n - \mathbf{P}_{(1)})\,\mathrm{d}\eta - \int_{-1}^{z} \mathbf{P}_{w(1)}\,\mathrm{d}\eta \right\} + \mathbf{J}(z), \quad (59)$$

where $\mathbf{u}_1^n$ and $\mathbf{w}_1^n$ are as yet unknown integration constants at node i. The lower limits in the integrals $\mathbf{G}$ and $\mathbf{H}$ are deliberately specified to be zero so that $\mathbf{w}_1^n$ and $\mathbf{u}_1^n$ represent the ε^2-order corrections to the midplane displacements, and

$$\mathbf{G}(z) = -\Phi_3^{-1}\int_0^z \mathbf{f}\mathbf{u}_{(0)}^n\,\mathrm{d}\eta, \quad \mathbf{H}(z) = \Phi_1^{-1}\int_0^z (\mathbf{g}\boldsymbol{\sigma}_{S(0)}^n - \Phi_2\mathbf{G})\,\mathrm{d}\eta,$$

$$\mathbf{I}(z) = (\Phi_1^{\mathrm{T}})^{-1}\int_{-1}^z (\mathbf{f}^{\mathrm{T}}\sigma_{z(0)}^n + \Psi\mathbf{H})\,\mathrm{d}\eta, \quad \mathbf{J}(z) = (\Phi_3^{\mathrm{T}})^{-1}\Phi_2^{\mathrm{T}}(\Phi_1^{\mathrm{T}})^{-1}\int_{-1}^z (z-\eta)(\mathbf{f}^{\mathrm{T}}\sigma_{z(0)}^n + \Psi\mathbf{H})\,\mathrm{d}\eta.$$

Imposing the associated lateral boundary conditions on (58) and (59), we obtain

$$\int_{-1}^1 \Psi(\mathbf{u}_1^n - \eta\Phi_1^{-1}\Phi_2\mathbf{w}_1^n)\,\mathrm{d}\eta = -\int_{-1}^1 (\mathbf{f}^{\mathrm{T}}\sigma_{z(0)}^n + \Psi\mathbf{H} - \mathbf{P}_{(1)})\,\mathrm{d}\eta, \tag{60}$$

$$\Phi_2^{\mathrm{T}}(\Phi_1^{\mathrm{T}})^{-1}\int_{-1}^1 (1-\eta)[\Psi(\mathbf{u}_1^n - \eta\Phi_1^{-1}\Phi_2\mathbf{w}_1^n) - \mathbf{P}_{(1)}]\,\mathrm{d}\eta - \int_{-1}^1 \mathbf{P}_{w(1)}\,\mathrm{d}\eta$$
$$= -\Phi_2^{\mathrm{T}}(\Phi_1^{\mathrm{T}})^{-1}\int_{-1}^1 (1-\eta)(\mathbf{f}^{\mathrm{T}}\sigma_{z(0)}^n + \Psi\mathbf{H})\,\mathrm{d}\eta. \tag{61}$$

Equations (60) and (61) are of the same form as (46) and (47). Therefore we can write out at once the system equations at the ε^2-order level:

$$\begin{bmatrix}\mathbf{K}_{uu} & \mathbf{K}_{uw}\\ \mathbf{K}_{wu} & \mathbf{K}_{ww}\end{bmatrix}\begin{Bmatrix}\mathbf{u}_1^n\\ \mathbf{w}_1^n\end{Bmatrix} = \begin{Bmatrix}\mathbf{F}_{u1}\\ \mathbf{F}_{w1}\end{Bmatrix}, \tag{62}$$

where

$$\mathbf{F}_{u1} = -\int_{-1}^1 (\mathbf{f}^{\mathrm{T}}\sigma_{z(0)}^n + \Psi\mathbf{H} - \mathbf{P}_{(1)})\,\mathrm{d}z,$$

$$\mathbf{F}_{w1} = \Phi_2^{\mathrm{T}}(\Phi_1^{\mathrm{T}})^{-1}\int_{-1}^1 z(\mathbf{f}^{\mathrm{T}}\sigma_{z(0)}^n + \Psi\mathbf{H} - \mathbf{P}_{(1)})\,\mathrm{d}z + \int_{-1}^1 \mathbf{P}_{w(1)}\,\mathrm{d}z.$$

The stiffness matrix is the same as the one at the leading-order level and the generalized force vector is known from the leading-order solution. Solution of (62) gives the corrections to the nodal values of the midplane displacements. Substituting these values into (56)–(59), we obtain the ε^2-order corrections to the nodal displacements and transverse stresses. The corrections to the in-plane stresses are determined using

$$\begin{Bmatrix}\sigma_{x(1)}\\ \sigma_{y(1)}\\ \sigma_{xy(1)}\end{Bmatrix} = \begin{bmatrix}d_{14} & d_{24}\\ d_{15} & d_{25}\\ d_{16} & d_{26}\end{bmatrix}\begin{Bmatrix}u_{(1)}\\ v_{(1)}\end{Bmatrix} + \begin{Bmatrix}\tilde{c}_{13}\\ \tilde{c}_{23}\\ \tilde{c}_{36}\end{Bmatrix}\sigma_{z(0)}. \tag{63}$$

The asymptotic solution can be carried on to higher orders in a similar way. At the k-th level the relevant equations are of the same forms as (56)–(63) except that

$$\mathbf{G}_k(z) = -\Phi_3^{-1}\int_0^z (\mathbf{f}\mathbf{u}_{(k-1)}^n - \mathbf{h}\tilde{c}_{33}^{-1}\sigma_{z(k-2)}^n)\,\mathrm{d}z, \quad \mathbf{H}_k(z) = \Phi_1^{-1}\int_0^z (\mathbf{g}\boldsymbol{\sigma}_{S(k-1)}^n - \Phi_2\mathbf{G}_k)\,\mathrm{d}z,$$

in which

$$\mathbf{h} = \int_{\Omega_e} \mathbf{N}_z^{\mathrm{T}}\mathbf{N}_z\,\mathrm{d}x\,\mathrm{d}y.$$

5. APPLICATION TO BENCHMARK PROBLEM

The elasticity solution obtained by Pagano (1970) for the problem of a simply supported multilayered composite plate under sinusoidal loads is generally regarded as a benchmark for verifying numerical models. The asymptotic finite element model is now examined by applying it to this problem.

Let us consider a rectangular laminated plate composed of orthotropic layers with the lamination such that

$$A_{16} = A_{26} = D_{16} = D_{26} = B_{16} = B_{26} = 0. \tag{64}$$

The plate is subject to the lateral load represented by

$$\bar{q}^{-}(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \alpha x \sin \beta y, \tag{65}$$

where $\alpha = m\pi/a$ and $\beta = n\pi/b$. We shall drop the double summation signs and assume that the field variables have been made dimensionless according to (14) for brevity.

The simple support edge conditions are of a mixed type given by

$$\begin{aligned} \sigma_x = v = w = 0 \quad &\text{on} \quad x = 0, a, \\ \sigma_y = u = w = 0 \quad &\text{on} \quad y = 0, b. \end{aligned} \tag{66}$$

For this problem it is advantageous to employ the following special functions to represent the variations of the displacements and transverse stresses in the domain:

$$N_{ui} = N_{xzi} = \cos \alpha x \sin \beta y, \tag{67}$$

$$N_{vi} = N_{yzi} = \sin \alpha x \cos \beta y, \tag{68}$$

$$N_{wi} = N_{zi} = \sin \alpha x \sin \beta y. \tag{69}$$

The displacements corresponding to (67)–(69) satisfy the displacement boundary conditions in (66). Using them in the leading-order expressions, and performing integration over the domain $0 < x < a$ and $0 < y < b$, we obtain the relevant matrices needed in the leading-order solution (see Appendix). Substituting these matrices into (48) and imposing the traction boundary conditions on the generalized force vector, we obtain

$$\begin{bmatrix} A_{11}\alpha^2 + A_{66}\beta^2 & \alpha\beta(A_{12}+A_{66}) & -\alpha^3 B_{11} - \alpha\beta^2(B_{12}+2B_{66}) \\ & A_{66}\alpha^2 + A_{22}\beta^2 & -\alpha^2\beta(B_{12}+2B_{66}) - \beta^3 B_{22} \\ \text{symmetric} & & \alpha^4 D_{11} + 2\alpha^2\beta^2(D_{12}+2D_{66}) + \beta^4 D_{22} \end{bmatrix} \begin{Bmatrix} u_{0i} \\ v_{0i} \\ w_{0i} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q_{mn} \end{Bmatrix}. \tag{70}$$

Solving (70) for u_{0i}, v_{0i} and w_{0i} and substituting these values into relevant expressions, we obtain the leading-order displacements and stresses. The results are given in the Appendix.

Let us proceed to ε^2 order to obtain modifications to the leading-order approximation. The finite element equations at the ε^2-order level are obtained by substituting the leading-order solution into the relevant expressions and determining the generalized force vector. The equations are

$$\frac{ab}{4}\begin{bmatrix} A_{11}\alpha^2+A_{66}\beta^2 & \alpha\beta(A_{12}+A_{66}) & -\alpha^3 B_{11}-\alpha\beta^2(B_{12}+2B_{16}) \\ & A_{66}\alpha^2+A_{22}\beta^2 & -\alpha^2\beta(B_{12}+2B_{16})-\beta^3 B_{22} \\ \text{symmetric} & & \alpha^4 D_{11}+2\alpha^2\beta^2(D_{12}+2D_{66})+\beta^4 D_{22} \end{bmatrix} \begin{Bmatrix} u_{1i} \\ v_{1i} \\ w_{1i} \end{Bmatrix} = \begin{Bmatrix} F_{u1i} \\ F_{v1i} \\ F_{w1i} \end{Bmatrix}. \quad (71)$$

After solving (71) for u_{1i}, v_{1i} and w_{1i}, the ε^2-order modifications are easily obtained. The results and the relevant matrices are given in the Appendix. The displacements and stresses after two steps of the asymptotic solution are given by combining the leading-order solution and the higher-order corrections according to (17). The solution can be carried on to higher orders with straightforward computations.

While the asymptotic solution to the problem has been determined following the numerical procedure, with the special interpolation functions (67)–(69) the whole domain is in effect treated as a single element. When polynomial functions are used for interpolation, it is necessary to discretize the x–y plane domain into finite elements and compute the stiffness matrix and the generalized force vector numerically. As mentioned earlier, the admissible functions for the transverse stresses require C^0 continuity and those for the displacements should be continuous in each element and conforming on interelement boundaries. We remark that the asymptotic finite element formulation is in some way a mixed model. In the mixed formulation, the issue of suppression of the spurious zero energy deformation modes has been raised (Pian and Chen, 1983). Special care must be taken in selecting proper interpolation functions in this regard. A continuing study on the computational aspects of the proposed approach is required.

The performance of the asymptotic solution is now demonstrated by computing the results for a four layer [0/90/90/0] rectangular laminated plate ($b = 3a$) under sinusoidal lateral loads. The layer material properties are given by $E_L = 25E_T$, $G_{LT} = 0.5E_T$, $G_{TT} = 0.2E_T$, $\nu_{LT} = \nu_{TT} = 0.25$, where the subscripts $_L$ and $_T$ refer to the longitudinal direction and transverse to fiber direction. The elastic constants can be deduced from these data. In Table 1 numerical results for the transverse displacement and the stresses at several selected locations, compared with Pagano's solution and the CLT results are presented. The transverse displacement and the stresses are normalized as $\bar{w} = 100E_T w/q_{11}aS^4$, $[\bar{\sigma}_x, \bar{\sigma}_y, \bar{\sigma}_{xy}] = [\sigma_x, \sigma_y, \sigma_{xy}]/q_{11}S^2$, $[\bar{\sigma}_{xz}, \bar{\sigma}_{yz}] = [\sigma_{xz}, \sigma_{yz}]/q_{11}S$, $\bar{\sigma}_z = \sigma_z/q_{11}$, in which $S = a/2h$. When $a/2h > 20$ ($\varepsilon < 0.025$) the asymptotic solution at the ε^2-order level are practically the same as the elasticity solution. As the thickness increases, the higher-order corrections become significant. When $a/2h < 20$, the thickness effect of the plate is significant. It is necessary to carry out the solution to higher orders so as to obtain more accurate results. In any case, carrying out two steps of the asymptotic solution is sufficient to produce acceptable results. To have a general idea of the performance of the asymptotic solution, we show in Figs 1–9 the variations of the displacements and the stresses through the thickness at selected points of a thick laminated plate ($S = 4$). The heterogeneous effect through the thickness is very pronounced in this case. The CLT solution is totally unacceptable, whereas the asymptotic solution approaches to the exact solution rapidly. With three steps of the computation, the ε^4-order solution yields very accurate interlaminar stresses as well as displacements through the thickness.

6. CONCLUSIONS

An asymptotic variational model for anisotropic inhomogeneous and laminated plates has been developed within the framework of three-dimensional elasticity. By means of asymptotic expansion the Hellinger–Reissner functional for the problem is decomposed into a sequence of functionals of various orders, which in turn can be used in conjunction with the finite element method to determine hierarchically the three-dimensional solution for the problem. The formulation begins with the displacements and transverse stresses as

Table 1. Comparisons of the results for a [0°/90°/90°/0°] laminated plate

$a/2h$	Theories	$\bar{w}$ $(a/2, b/2, 0)$	$\bar{\sigma}_x$ $(a/2, b/2, -1)$	$\bar{\sigma}_y$ $(a/2, b/2, -1/2)$	$\bar{\sigma}_{xy}$ $(0, 0, -1)$	$\bar{\sigma}_{xz}$ $(0, b/2, 0)$	$\bar{\sigma}_{yz}$ $(a/2, 0, 0)$	$\bar{\sigma}_z$ $(a/2, b/2, 0)$
	Exact	3.2337	−1.2180	−0.1917	0.0307	0.3548	0.0501	−0.5008
	CLT	0.5504	−0.6815	−0.0412	0.0091	0.4093	0.0144	−0.5000
4	Present (ε^0)	0.5504	−0.6815	−0.0412	0.0091	0.4093	0.0144	−0.5000
	Present (ε^2)	3.5340	−1.3168	−0.2337	0.0358	0.3387	0.0605	−0.5039
	Present (ε^4)	3.1481	−1.1843	−0.1796	0.0291	0.3597	0.0472	−0.5002
	Exact	1.0183	−0.7799	−0.0707	0.0132	0.3985	0.0215	−0.5005
	CLT	0.5504	−0.6815	−0.0412	0.0091	0.4093	0.0144	−0.5000
10	Present (ε^0)	0.5504	−0.6815	−0.0412	0.0091	0.4093	0.0144	−0.5000
	Present (ε^2)	1.0278	−0.7831	−0.0720	0.0133	0.3980	0.0218	−0.5006
	Present (ε^4)	1.0179	−0.7797	−0.0706	0.0132	0.3985	0.0215	−0.5005
	Exact	0.6691	−0.7067	−0.0488	0.0101	0.4065	0.0163	−0.5002
20	CLT	0.5504	−0.6815	−0.0412	0.0091	0.4093	0.0144	−0.5000
	Present (ε^0)	0.5504	−0.6815	−0.0412	0.0091	0.4093	0.0144	−0.5000
	Present (ε^2)	0.6697	−0.7069	−0.0489	0.0101	0.4065	0.0163	−0.5002

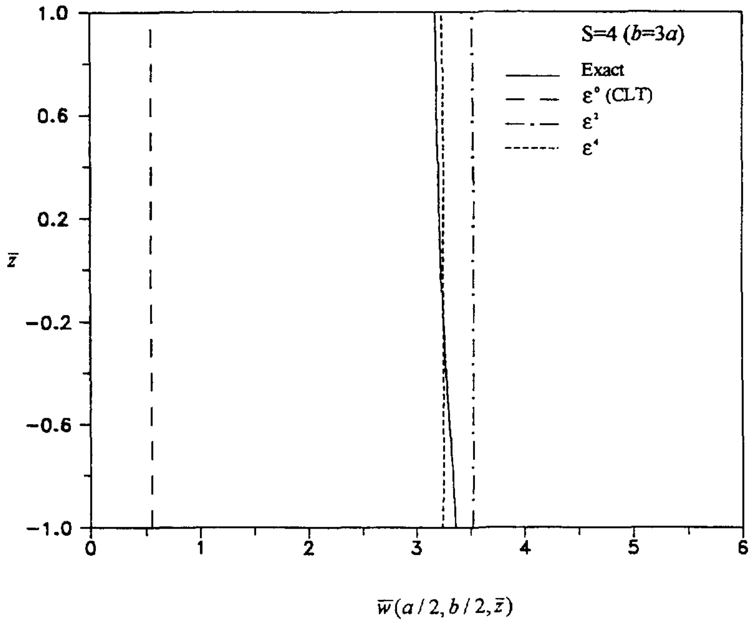

Fig. 1. Variation of the transverse displacement $\bar{w}$ through the thickness of a [0/90/90/0] laminated plate.

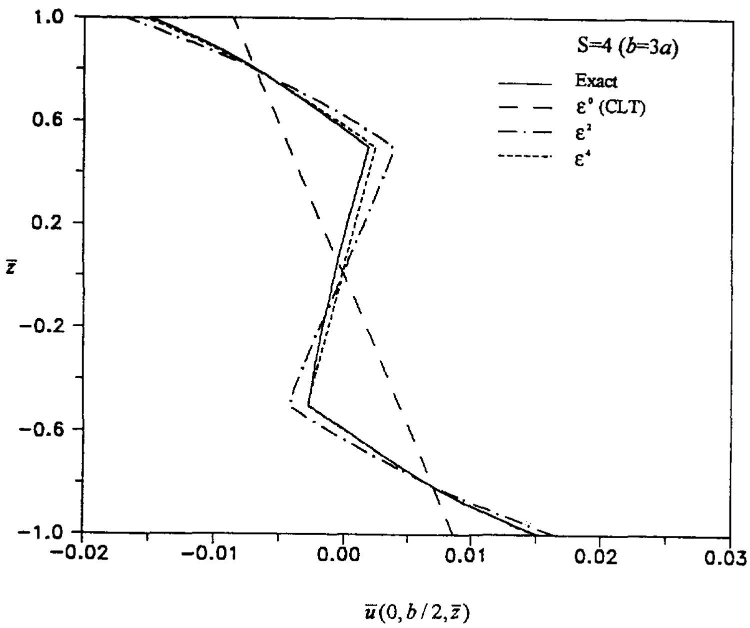

Fig. 2. Variation of the in-plane displacement $\bar{u}$ through the thickness of a [0/90/90/0] laminated plate.

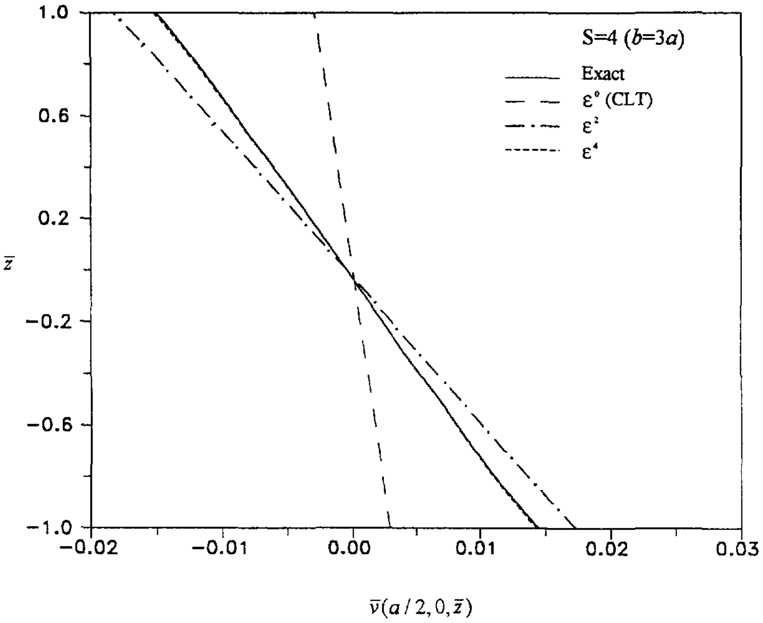

Fig. 3. Variation of the in-plane displacement $\bar{v}$ through the thickness of a [0/90/90/0] laminated plate.

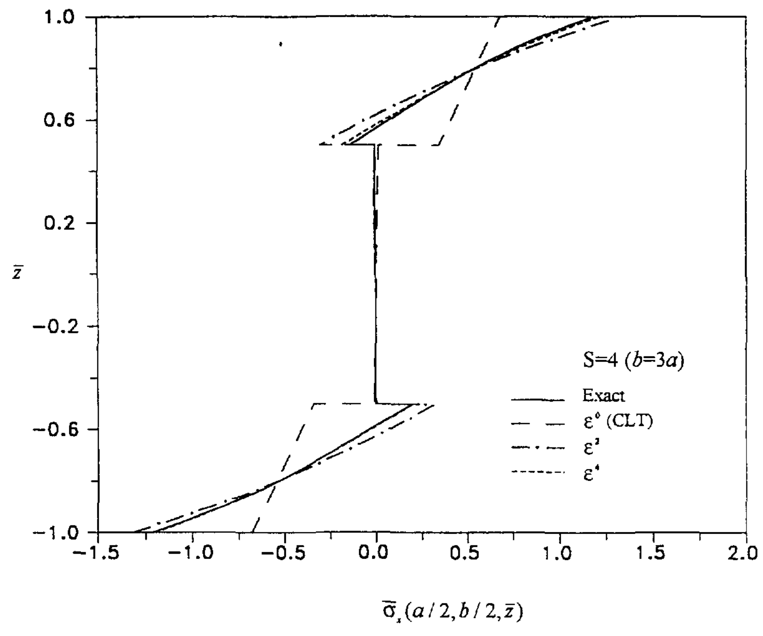

Fig. 4. Variation of the in-plane normal stress $\bar{\sigma}_x$ through the thickness of a [0/90/90/0] laminated plate.

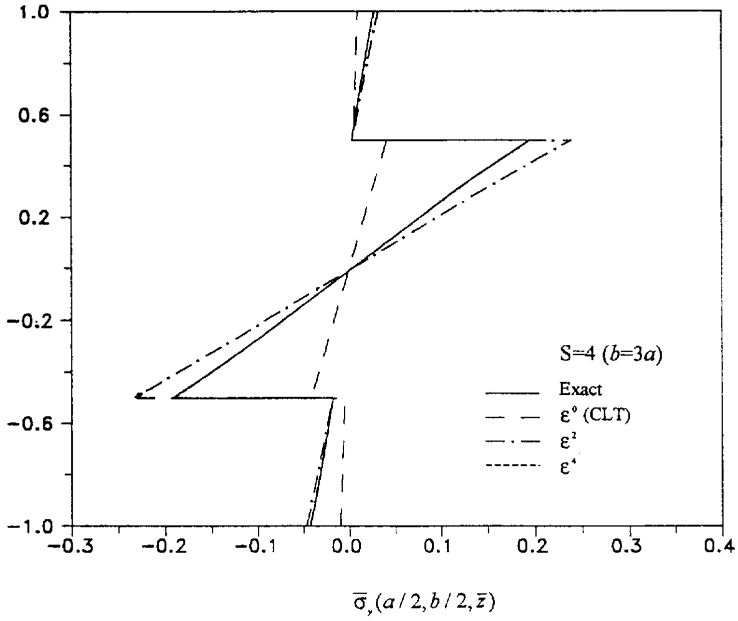

Fig. 5. Variation of the in-plane normal stress $\bar{\sigma}_y$ through the thickness of a [0/90/90/0] laminated plate.

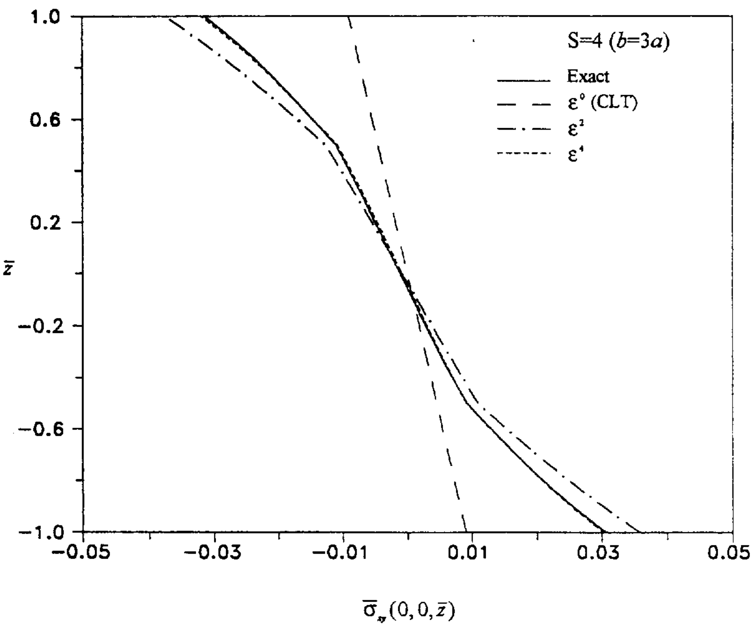

Fig. 6. Variation of the in-plane shearing stress $\bar{\sigma}_{xy}$ through the thickness of a [0/90/90/0] laminated plate.

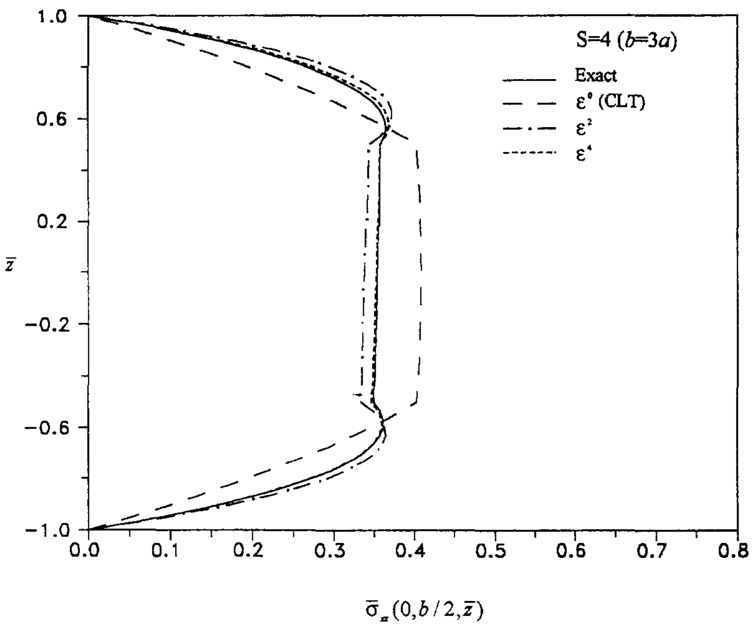

Fig. 7. Variation of the transverse shearing stress $\bar{\sigma}_{xz}$ through the thickness of a [0/90/90/0] laminated plate.

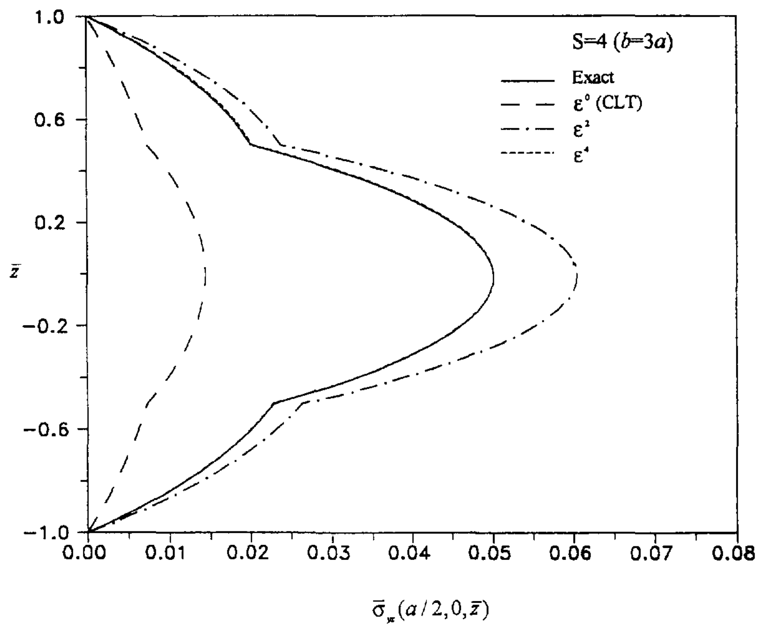

Fig. 8. Variation of the transverse shearing stress $\bar{\sigma}_{yz}$ through the thickness of a [0/90/90/0] laminated plate.

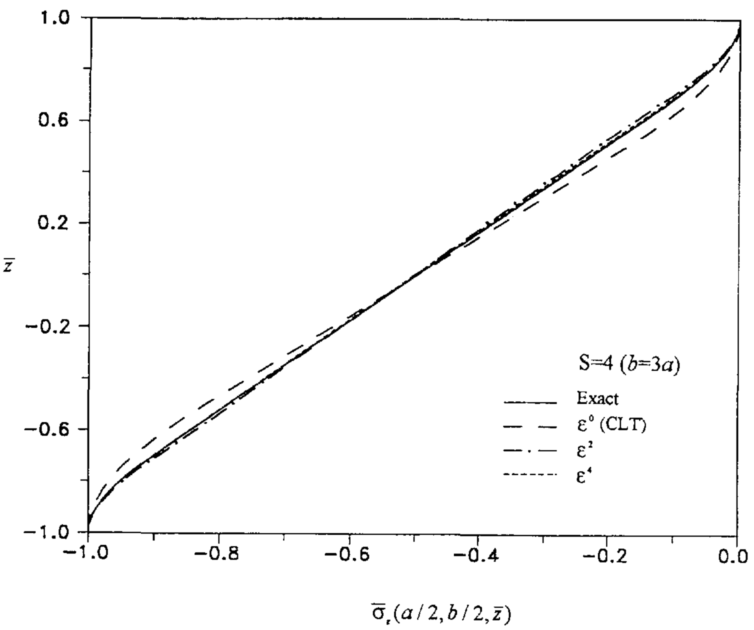

Fig. 9. Variation of the transverse normal stress $\bar{\sigma}_z$ through the thickness of a [0/90/90/0] laminated plate.

the functions subject to variation, but ends up with only three displacement DOF on the midplane as unknowns in the system equations. As a result, the number of DOF at each level is less than that of a homogeneous Kirchhoff plate in bending and stretching. In the multilevel computations, the stiffness matrix remains unchanged; the one generated at the leading-order level is always used at subsequent levels. The formulation is three-dimensional yet requires only two-dimensional interpolation on the plane form of the plate. There is no need of interpolation in the thickness direction even in the case of multilayered plates. The heterogeneous effect through the thickness is taken into account in a consistent and systematic way. Applications of the method to the benchmark problem show that it is effective in numerical modeling of multilayered composite plates.

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APPENDIX

The relevant matrices for the leading-order solution of the benchmark problem are

$$\mathbf{\Psi} = [\psi_{ii}],\quad \psi_{ii} = \frac{ab}{4}\begin{bmatrix}(\tilde{Q}_{11}\alpha^2+\tilde{Q}_{66}\beta^2) & \alpha\beta(\tilde{Q}_{12}+\tilde{Q}_{66}) \\ \text{symmetric} & (\tilde{Q}_{66}\alpha^2+\tilde{Q}_{22}\beta^2)\end{bmatrix},\quad \mathbf{\Phi}_1 = \frac{ab}{4}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix},\quad \mathbf{\Phi}_2 = \frac{ab}{4}\left[\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}_{ii}\right],$$

$$\mathbf{\Phi}_3 = \frac{ab}{4}\mathbf{I},\quad \tilde{\mathbf{A}} = [\mathbf{a}_{ii}],\quad \tilde{\mathbf{B}} = [\mathbf{b}_{ii}],\quad \tilde{\mathbf{D}} = [\mathbf{d}_{ii}],$$

$$\mathbf{a}_{ii} = \frac{ab}{4}\begin{bmatrix}(A_{11}\alpha^2+A_{66}\beta^2) & \alpha\beta(A_{12}+A_{66})\\ \text{symmetric} & (A_{66}\alpha^2+A_{22}\beta^2)\end{bmatrix},\quad \mathbf{b}_{ii} = \frac{ab}{4}\begin{bmatrix}(B_{11}\alpha^2+B_{66}\beta^2) & \alpha\beta(B_{12}+B_{66})\\ \text{symmetric} & (B_{66}\alpha^2+B_{22}\beta^2)\end{bmatrix},$$

$$\mathbf{d}_{ii} = \frac{ab}{4}\begin{bmatrix}(D_{11}\alpha^2+D_{66}\beta^2) & \alpha\beta(D_{12}+D_{66})\\ \text{symmetric} & (D_{66}\alpha^2+D_{22}\beta^2)\end{bmatrix},\quad \mathbf{K}_{uu} = \frac{ab}{4}\left[\begin{bmatrix}(A_{11}\alpha^2+A_{66}\beta^2) & \alpha\beta(A_{12}+A_{66})\\ \text{symmetric} & (A_{66}\alpha^2+A_{22}\beta^2)\end{bmatrix}_{ii}\right],$$

$$\mathbf{K}_{ww} = \frac{ab}{4}[[\alpha^4 D_{11}+2\alpha^2\beta^2(D_{12}+2D_{66})+\beta^4 D_{22}]_{ii}],\quad \mathbf{K}_{uw} = \mathbf{K}_{wu}^{\mathrm{T}} = -\frac{ab}{4}\left[\begin{Bmatrix}\alpha^3 B_{11}+\alpha\beta^2(B_{12}+2B_{66})\\ \alpha^2\beta(B_{12}+2B_{66})+\beta^3 B_{22}\end{Bmatrix}_{ii}\right],$$

where $\mathbf{I}$ is a unit matrix, the subscript ii indicates that only the diagonal submatrices are present and all others off the diagonal are null.

The leading-order solutions for the problem are

$$u_{(0)} = u_0(z)\cos\alpha x\sin\beta y, \tag{A1}$$

$$v_{(0)} = v_0(z)\sin\alpha x\cos\beta y, \tag{A2}$$

$$w_{(0)} = w_{0i}\sin\alpha x\sin\beta y, \tag{A3}$$

$$\sigma_{xz(0)} = \sigma_{xz0}(z)\cos\alpha x\sin\beta y, \tag{A4}$$

$$\sigma_{yz(0)} = \sigma_{yz0}(z)\sin\alpha x\cos\beta y, \tag{A5}$$

$$\sigma_{z(0)} = \sigma_{z0}(z)\sin\alpha x\sin\beta y, \tag{A6}$$

$$\sigma_{x(0)} = -[\alpha\tilde{Q}_{11}u_0(z)+\beta\tilde{Q}_{12}v_0(z)]\sin\alpha x\sin\beta y, \tag{A7}$$

$$\sigma_{y(0)} = -[\alpha\tilde{Q}_{12}u_0(z)+\beta\tilde{Q}_{22}v_0(z)]\sin\alpha x\sin\beta y, \tag{A8}$$

$$\sigma_{xy(0)} = [\beta\tilde{Q}_{66}u_0(z)+\alpha\tilde{Q}_{66}v_0(z)]\cos\alpha x\cos\beta y, \tag{A9}$$

in which $u_0(z) = u_{0i}-\alpha w_{0i}z$, $v_0(z) = v_{0i}-\beta w_{0i}z$,

$$\begin{Bmatrix}\sigma_{xz0}(z)\\ \sigma_{yz0}(z)\end{Bmatrix} = \int_{-1}^{z}\begin{Bmatrix}\varphi_{xz0}(\eta)\\ \varphi_{yz0}(\eta)\end{Bmatrix}\mathrm{d}\eta,$$

$$\varphi_{xz0}(\eta) = (\alpha^2\tilde{Q}_{11}+\beta^2\tilde{Q}_{66})u_0(\eta)+\alpha\beta(\tilde{Q}_{12}+\tilde{Q}_{66})v_0(\eta),$$

$$\varphi_{yz0}(\eta) = \alpha\beta(\tilde{Q}_{12}+\tilde{Q}_{66})u_0(\eta)+(\alpha^2\tilde{Q}_{66}+\beta^2\tilde{Q}_{22})v_0(\eta),$$

$$\sigma_{z0}(z) = \int_{-1}^{z}(z-\eta)[\alpha\varphi_{xz0}(\eta)+\beta\varphi_{yz0}(\eta)]\,\mathrm{d}\eta-q_{mn}.$$

The generalized force vectors at the ε^2-order level are given by

$$\mathbf{F}_{ui} = \begin{Bmatrix}F_{u1i}\\ F_{v1i}\end{Bmatrix} = -\frac{ab}{4}\int_{-1}^{1}\begin{Bmatrix}\phi_{xz}(\eta)\\ \phi_{yz}(\eta)\end{Bmatrix}\mathrm{d}\eta,\quad F_{w1i} = \frac{ab}{4}\int_{-1}^{1}\eta[\alpha\phi_{xz}(\eta)+\beta\phi_{yz}(\eta)]\,\mathrm{d}\eta,$$

$$\phi_{xz}(\eta) = (\alpha^2\tilde{Q}_{11}+\beta^2\tilde{Q}_{66})H_u(\eta)+\alpha\beta(\tilde{Q}_{12}+\tilde{Q}_{66})H_v(\eta)-\alpha\tilde{c}_{13}\sigma_{z0}(\eta),$$

$$\phi_{yz}(\eta) = \alpha\beta(\tilde{Q}_{12}+\tilde{Q}_{66})H_u(\eta)+(\alpha^2\tilde{Q}_{66}+\beta^2\tilde{Q}_{22})H_v(\eta)-\beta\tilde{c}_{23}\sigma_{z0}(\eta),$$

$$\begin{Bmatrix} H_u(z) \\ H_v(z) \end{Bmatrix} = \int_0^z \begin{Bmatrix} \tilde{s}_{55}\sigma_{xz0}(\eta) - \alpha G(\eta) \\ \tilde{s}_{44}\sigma_{yz0}(\eta) - \beta G(\eta) \end{Bmatrix} \mathrm{d}\eta,$$

$$G(\eta) = \int_0^z [\alpha\tilde{c}_{13}u_0(\eta) + \beta\tilde{c}_{23}v_0(\eta)]\,\mathrm{d}\eta.$$

The ε^2-order corrections for the problem are

$$u_{(1)} = [u_1(z) + H_u(z)]\cos\alpha x \sin\beta y, \tag{A10}$$

$$v_{(1)} = [v_1(z) + H_v(z)]\sin\alpha x \cos\beta y, \tag{A11}$$

$$w_{(1)} = [w_{1i} + G(z)]\sin\alpha x \sin\beta y, \tag{A12}$$

$$\sigma_{xz(1)} = [\sigma_{xz1}(z) + \int_{-1}^{z} \phi_{xz0}(\eta)\,\mathrm{d}\eta]\cos\alpha x \sin\beta y, \tag{A13}$$

$$\sigma_{yz(1)} = [\sigma_{yz1}(z) + \int_{-1}^{z} \phi_{yz0}(\eta)\,\mathrm{d}\eta]\sin\alpha x \cos\beta y, \tag{A14}$$

$$\sigma_{z(1)} = \{\sigma_{z1}(z) + \int_{-1}^{z} (z-\eta)[\alpha\phi_{xz0}(\eta) + \beta\phi_{yz0}(\eta)]\,\mathrm{d}\eta\}\sin\alpha x \sin\beta y, \tag{A15}$$

$$\sigma_{x(1)} = -[\alpha\tilde{Q}_{11}u_1(z) + \beta\tilde{Q}_{12}v_1(z) - \tilde{c}_{13}\sigma_{z0}(z)]\sin\alpha x \sin\beta y, \tag{A16}$$

$$\sigma_{y(1)} = -[\alpha\tilde{Q}_{12}u_1(z) + \beta\tilde{Q}_{22}v_1(z) - \tilde{c}_{23}\sigma_{z0}(z)]\sin\alpha x \sin\beta y, \tag{A17}$$

$$\sigma_{xy(1)} = -[\beta\tilde{Q}_{66}u_1(z) + \alpha\tilde{Q}_{66}v_1(z)]\cos\alpha x \cos\beta y, \tag{A18}$$

in which $u_1(z) = u_{1i} - \alpha w_{1i}z$, $v_1(z) = v_{1i} - \beta w_{1i}z$, and

$$\begin{Bmatrix} \sigma_{xz1}(z) \\ \sigma_{yz1}(z) \end{Bmatrix} = \int_{-1}^{z} \begin{Bmatrix} \varphi_{xz1}(\eta) \\ \varphi_{yz1}(\eta) \end{Bmatrix} \mathrm{d}\eta,$$

$$\varphi_{xz1}(\eta) = (\alpha^2\tilde{Q}_{11} + \beta^2\tilde{Q}_{66})u_1(\eta) + \alpha\beta(\tilde{Q}_{12} + \tilde{Q}_{66})v_1(\eta),$$

$$\varphi_{yz1}(\eta) = \alpha\beta(\tilde{Q}_{12} + \tilde{Q}_{66})u_1(\eta) + (\alpha^2\tilde{Q}_{66} + \beta^2\tilde{Q}_{22})v_1(\eta),$$

$$\sigma_{z1}(z) = \int_{-1}^{z} (z-\eta)[\alpha\varphi_{xz1}(\eta) + \beta\varphi_{yz1}(\eta)]\,\mathrm{d}\eta.$$